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" Long Period Coupling Terms

For Lagrange's Equations "

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LONG PERIOD COUPLING TERMS IN LAGRANGE'S EQUATIONS

Introduction

In this report the long period terms arising from the short-short period coupling of zonal harmonics are derived for Lagrange's Equations. The formulation is general so that the results are valid for any pairs of zonal harmonics $J_n - J_\ell$ where n and ℓ are arbitrary.

Formulas are given to generate the various functions and integrals needed for the results given in this report. Checks have been made against the work of Kozai, Reference 1.

This paper is a generalization of that portion of the work of Berger, Reference 2, which deals with the long period coupling effect of certain pairs of zonal harmonics.

Analysis

Lagrange's Equations

The equations of Lagrange are:

$$\frac{da}{dt} = \frac{2a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \frac{\partial R}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \frac{\partial R}{\partial \omega} \qquad (1)$$

$$\frac{di}{dt} = \frac{\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \sqrt{1-e^2}} \frac{\partial R}{\partial \omega}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \sqrt{1-e^2}\, \sin i} \frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \frac{\partial R}{\partial e}$$

$$\frac{d\Omega}{dt} = \frac{1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \sqrt{1-e^2}\, \sin i} \frac{\partial R}{\partial i}$$

$$\frac{dM}{dt} = \mu^{\frac{1}{2}} a^{-\frac{3}{2}} - \frac{1-e^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \frac{\partial R}{\partial e} - \frac{2a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} \frac{\partial R}{\partial a}$$

The Disturbing Function

The short period disturbing function is given by

$$R = \sum_{n=2}^{N} R_n$$

where

$$R_n = \sum_{i=0}^{\frac{n-ODN}{2}} \frac{\mu J_n}{a^{n+1}} B_{n,q} \left[\left(\frac{a}{r}\right)^{n+1} \cos(\alpha + q f) - a_{n-1,q} \cos\alpha \right] \qquad (2)$$

$$q = 2i + ODN$$

where $ODN = 0$ for n even

$= 1$ for n odd

$$\alpha = q\omega - ODN \frac{\pi}{2}$$

This notation is introduced to keep the form of the disturbing function invariant. This we do not need to write separate equations for R_n for the cases n even or odd.

Since we deal with coupling of zonals n and ℓ the function R_ℓ is defined similarly to R_n were n, q, i, ODN and α are replaced by ℓ, s, j, ODL and β, respectively.

The quantities $B_{n,q}$ and $a_{n-1,q}$ are the inclination and averaging functions respectively and are given in Appendix 1.

The product $a_{n-1,q} \cos\alpha$ in Equation 1, is the average of the term $\left(\frac{a}{r}\right)^{n+1} \cos(\alpha + q f)$ taken over the mean anomaly.

Expansions For Lagrange's Equations

The Taylor series expansion for a typical Keplerian element E_i needed to find the coupling terms in Equations (1) is

$$\frac{dE_i}{dt} = \sum_{n=2}^{N} (f_i) + \left[\sum_{n=2}^{N} \sum_{l=2}^{L} \left\{ \left(a \frac{\partial f_i}{\partial a} \right)_n \left(\frac{\delta a}{a} \right)_l + \left(\frac{\partial f_i}{\partial e} \right)_n (\delta e)_l + \left(\frac{\partial f_i}{\partial i} \right)_n (\delta i)_l \right. \right. \tag{3}$$

$$\left. + \left(\frac{\partial f_i}{\partial \omega} \right)_n (\delta \omega)_l + \left(\frac{\partial f_i}{\partial M} \right)_n (\delta M)_l \right]$$

f_i represents the functions appearing on the right hand side of Equations (1) for the i^{th} element. For example for the inclination

$$f_i = \frac{\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin i} \frac{\partial R_n}{\partial \omega}$$

The symbol δ attached to an element indicates the perturbations in that element. The subscripts n and l pertain to the zonal harmonics considered.

General formulas for f_i, $\frac{\partial f_i}{\partial E_j}$, and for ∇E_i are given in appendices 3 and 4.

In Equation (3), (N-1)(L-1) coupling terms arise. Formulas for the general $J_n - J_\ell$ long period term are derived by averaging the quantities enclosed by the brackets in Equation (3), using the functions defined by the appendices. The results are functions of α and β which are defined in connection with Equation (2).

It should be noted that long period perturbations due to $J_n - J_\ell$ coupling do not occur in the semi major axis. This was proved by Kozai, Reference 4. His proof while given only for $J_2 - J_2$ coupling can be readily extended for 2 nd order coupling of the form $J_n - J_\ell$. Long period terms in the semi major axis "a" result from higher order coupling of the form $J_n - J_\ell - J_m$, which is beyond the scope of this report. However for the special but important case for long period terms of order J_2^2 the results of Berger, Reference 5, are available.

Evaluation of the Expansions For Lagrange's Equations

The mean values of the functions enclosed by the brackets in Equation (3) are now given for each of the Keplerian elements under consideration. Since certain types of trigonometric products recur frequently they are given in Appendix 5.

In general the functions to be averaged are of the form

$$\left(\frac{a}{r}\right)^{p+1} \cos(\alpha + q f)$$

so that the average values are readily [illegible] by means of the averaging function, $a_{p-1,q}$ described in Appendix 1.

When the subscripts of the averaging function are functions of a parameter e.g. $a_{\nu(t),\phi(t)}$, then it is useful to find the maximum value of t, t_{max} , which will yield non-zero terms for the averaging function. Then t_{max} is found from the relation

$$\nu(t) - |\phi(t)| = 0$$

For example if $\nu(t) = n$, $\phi(t) = q - 1 - t$, then

$$t_{max} = n + q - 1$$

In some of the series a term occurs in which $s - k$ is a divisor, where k is a summation index. These series have been derived so that the correct sum is obtained by omitting the term for which $s - k = 0$.

The Inclination Equation

The expansion for the inclination equation corresponding to Equation (3) is given by

$$\frac{di}{dt} = \left(f_3 \frac{\partial R}{\partial \omega} \right) + \left[f_3 \left\{ \left(a \frac{\partial^2 R}{\partial \omega \partial a} \right)_n \left(\frac{\delta a}{a} \right)_\ell + \left(\frac{\partial^2 R}{\partial \omega \partial e} \right)_n (\delta e)_\ell + \left(\frac{\partial^2 R}{\partial \omega \partial i} \right)_n (\delta i)_\ell + \left(\frac{\partial^2 R}{\partial \omega^2} \right)_n (\delta \omega)_\ell \right\} + \left(\frac{\partial R}{\partial \omega} \right)_n \left\{ a \frac{\partial f_3}{\partial a} \left(\frac{\delta a}{a} \right)_\ell + \frac{\partial f_3}{\partial e} (\delta e)_\ell + \frac{\partial f_3}{\partial i} (\delta i)_\ell \right\} \right]$$

$$f_3 = \frac{\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin i}$$

$$a \frac{\partial f_3}{\partial a} = - \frac{\cos i}{2 \mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin i}$$

$$\frac{\partial f_3}{\partial e} = \frac{e \cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta^3 \sin i}$$

$$\frac{\partial f_3}{\partial i} = - \frac{1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin^2 i}$$

$$\eta = \sqrt{1 - e^2}$$

Using the functions given in the appendices,

$$1. \quad \overline{\left(a \frac{\partial^2 R}{\partial \omega \partial a} \quad \frac{\delta a}{a} \right)} = \frac{\mu J_n J_l}{a^{n+l+1}} \sum_q \sum_s 2(n+1) q \, B_{l,s} \, B_{n,q} \left(\overline{A'_n A_l} - \bar{A}'_n \bar{A}_l \right)$$

The bars over the functions indicate averages over the mean anomaly and consequently define the long period terms.

$\sum\limits_q$, $\sum\limits_s$ are abbreviations for $\sum\limits_{i=0}^{\frac{n-ODN}{2}}$, $\sum\limits_{j=0}^{\frac{l-ODL}{2}}$ respectively,
where we recall that

$$q = 2i + ODN \qquad i = 0, 1, 2, \ldots \frac{n - ODN}{2}$$

$$s = 2j + ODL \qquad j = 0, 1, 2, \ldots \frac{l - ODL}{2}$$

$$\alpha = q\omega - ODN \frac{\pi}{2}$$

$$\beta = s\omega - ODL \frac{\pi}{2}$$

The functions of the type A_n, A'_n etc. are defined in appendix 2.

$$\overline{A'_n A_l} - \overline{A'_n}\,\overline{A'_l} = \frac{1}{2}\Big\{\sin(\alpha+\beta)\big[a_{n+l,q+s} - a_{n-1,q}\,a_{l-1,s}\big] + \sin(\alpha-\beta)\big[a_{n+l,q-s} - a_{n-1,q}\,a_{l-1,s}\big]\Big\}$$

2. $$\overline{\frac{\partial^2 R}{\partial\omega\,\partial e}\,(\delta e)_1} = -\frac{\mu\,\eta^2 J_n J_l}{e\,a^{n+l+1}} \sum_q \sum_s q\, B_{n,q}\, B_{l,s}\left[\overline{F'_n A_l} - \overline{F'_n}\,\overline{A_l}\right]$$

$$\overline{F'_n A_l} = \sum_{d=1}^{4} \frac{c(d)}{2}\Big\{\sin(\alpha+\beta)\,(a_{\nu(d),\phi(d)}) + \sin(\alpha-\beta)\,(a_{\nu(d),\rho(d)})\Big\}$$

The parameters for $\overline{F'_n A_l}$ are given in Table 1.

TABLE 1 (Parameters for $\overline{F'_n A_l}$)

d	$c(d)$	$\nu(d)$	$\phi(d)$	$\rho(d)$
1	$\frac{n+1+q}{2}$	$n+l+1$	$q+s+1$	$q-s+1$
2	$\frac{q}{2\eta^2}$	$n+l$	$q+s+1$	$q-s+1$
3	$\frac{n+1-q}{2}$	$n+l+1$	$q+s-1$	$q-s-1$
4	$-\frac{q}{2\eta^2}$	$n+l$	$q+s-1$	$q-s-1$

$$\overline{F'_n}\,\overline{A_l} = \sin\alpha\cos\beta\left\{b_{n-1,q} - \frac{(2n-1)}{\eta}\,a_{n-1,q}\right\} a_{l-1,s}$$

$$(3)\quad \overline{\frac{\partial^2 R}{\partial\omega\,\partial e}(\delta e)_2} = \frac{-\mu\eta\, J_n J_e}{e\, a^{n+\ell+1}} \sum_q \sum_s s\, a_{\ell,s}\, B_{n,q}\, B_{\ell,s} \left(\overline{F_n' B_{KP}}\right) \sin\beta$$

$$\overline{F_n' B_{KP}} = \sum_{d=1}^{4} c(d) \left\{ \sum_{t=1}^{t_1} \gamma_t\, a_{\nu(d),\phi(d)} - \sum_{t=1}^{t_2} \gamma_t\, a_{\nu(d),\rho(d)} \right\}$$

The parameters for $\overline{F_n' B_{KP}}$ are given in Table 2.

TABLE 2 (Parameters for $\overline{F_n' B_{KP}}$)
($c(d)$ same as in Table 1)

d	$\nu(d)$	$\phi(d)$	$\rho(d)$	t_1	t_2
1	n	$q+1+t$	$q+1-t$	$n-q-1$	$n+q+1$
2	$n-1$	$q+1+t$	$q+1-t$	$n-q-2$	$n+q$
3	n	$q-1+t$	$q-1-t$	$n-q+1$	$n+q-1$
4	$n-1$	$q-1+t$	$q-1-t$	$n-q$	$n+q-2$

$$(4)\quad \overline{\frac{\partial^2 R}{\partial\omega\,\partial e}(\delta e)_3} = \frac{\mu\eta\, J_n J_e}{e\, a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} e_k\, s\, a_{\ell-1,k}\, B_{n,q}\, B_{\ell,s} \left(\overline{F_n' C_\ell} - \overline{F_n' C_\ell}\right)$$

$$\overline{F_n' C_\ell} = \sum_{d=1}^{4} \frac{c(d)}{2} \left\{ a_{\nu(d),\phi(d)} \frac{\sin(\alpha+\beta)}{s+k} + a_{\nu(d),\rho(d)} \frac{\sin(\alpha-\beta)}{s+k} \right.$$

$$\left. + a_{\nu(d),\sigma(d)} \frac{\sin(\alpha+\beta)}{s-k} + a_{\nu(d),\tau(d)} \frac{\sin(\alpha-\beta)}{s-k} \right\}$$

The parameters for $\overline{F_n' C_\ell}$ are given in Table 3.

TABLE 3 (Parameters for $\overline{F_n' C_\ell}$)
($c(d)$ same as in Table 1)

d	$\nu(d)$	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	n	$q+1+s+k$	$q+1-s-k$	$q+1+s-k$	$q+1-s+k$
2	$n-1$	$q+1+s+k$	$q+1-s-k$	$q+1+s-k$	$q+1-s+k$
3	n	$q-1+s+k$	$q-1-s-k$	$q-1+s-k$	$q-1-s+k$
4	$n-1$	$q-1+s+k$	$q-1-s-k$	$q-1+s-k$	$q-1-s+k$

$$\overline{F'_n \bar{C}_\ell} = \sin\alpha \cos\beta \, (\gamma_{s+k} + \gamma_{s-k})$$

5. $$\overline{\frac{\partial^2 R}{\partial \Omega \partial i}} (\delta i)_1 = - \frac{\mu J_n J_\ell \cos i}{\eta a^{n+\ell+1} \sin i} \sum_q \sum_s q^2 s \, a_{\ell-1,s} B_{n,q} B_{\ell,s} \ldots (\overline{A'_n B_{K\ell}})$$

$$\overline{A'_n B_{K\ell}} = \cos\alpha \left\{ \sum_{t=1}^{n-q-1} \gamma_t a_{n-1,q+t} - \sum_{t=1}^{n+q-1} \gamma_t a_{n-1,q-t} \right\}$$

6. $$\overline{\frac{\partial^2 R}{\partial \omega \partial i}} (\delta i)_2 = - \frac{\mu J_n J_\ell \cos i}{\eta a^{n+\ell+1} \sin i} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k q^2 s \, a_{\ell-1,k} B_{n,q} C_{\ell,s} (\overline{A'_n C_\ell} - \overline{\bar{A}'_n \bar{C}_\ell})$$

$$\overline{A'_n C_\ell} = \frac{1}{2} \left\{ \frac{\sin(\alpha+\beta)}{s+k} a_{n-1,q+s+k} + \frac{\sin(\alpha-\beta)}{s+k} a_{n-1,q-s-k} + \frac{\sin(\alpha+\beta)}{s-k} a_{n-1,q+s-k} \right.$$

$$\left. + \frac{\sin(\alpha-\beta)}{s-k} a_{n-1,q-s+k} \right\}$$

$$\overline{\bar{A}'_n \bar{C}_\ell} = \sin\alpha \cos\beta \, a_{n-1,q} \, a_{\ell-1,k} (\gamma_{s+k} + \gamma_{s-k})$$

7. $$\overline{\frac{\partial^2 R}{\partial \omega^2}} \overline{(\delta \omega)_1} = - \frac{\mu J_n J_\ell}{\eta^{2\ell} a^{n+\ell+1}} \sum_q \sum_s q^2 B_{n,q} W \cos\beta \, (\overline{A_n B_{K\ell}})$$

$$\overline{A_n B_{K\ell}} = - \sin\alpha \left\{ \sum_{t=1}^{n-q-1} \gamma_t a_{n-1,q+t} - \sum_{t=1}^{n+q-1} \gamma_t a_{n-1,q-t} \right\}$$

8. $$\overline{\frac{\partial^2 R}{\partial \omega^2}} (\delta \omega)_2 = \frac{-\mu J_n J_\ell}{\eta^{2\ell} a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k q^2 B_{n,q} W (\overline{A_n C'_\ell} - \overline{\bar{A}_n \bar{C}'_\ell})$$

$$\overline{A_n C'_\ell} = \frac{1}{2} \left\{ \frac{\sin(\alpha+\beta)}{s+k} a_{n-1,q+s+k} - \frac{\sin(\alpha-\beta)}{s+k} a_{n-1,q-s-k} + \frac{\sin(\alpha+\beta)}{s-k} a_{n-1,q+s-k} \right.$$

$$\left. - \frac{\sin(\alpha-\beta)}{s-k} a_{n-1,q-s+k} \right\}$$

$$\overline{\bar{A}_n \bar{C}'_\ell} : \sin\beta \cos\alpha \, (\gamma_{s+k} + \gamma_{s-k})$$

9. $$\frac{\partial^2 R}{\partial \omega^2}(\delta\omega)_3 = -\frac{\mu J_n J_\ell}{2 e a^{n+\ell+1}} \sum_q \sum_s q^2 B_{n,q} B_{\ell,s} \left(\overline{A_n D'_{1\ell}} - \bar{A}_n \bar{D}'_{1\ell}\right)$$

$$\overline{A_n D'_{1\ell}} = \frac{\sin(\alpha+\beta)}{2}\left[a_{n+\ell-1,s+1+q} - a_{n+\ell-1,s-1+q}\right] + \frac{\sin(\beta-\alpha)}{2}\left[a_{n+\ell-1,s+1-q} - a_{n+\ell-1,s-1-q}\right]$$

$$\bar{A}_n \bar{D}'_{1\ell} = \sin\beta\cos\alpha\left[a_{\ell-2,s+1} - a_{\ell-2,s-1}\right] a_{n-1,q}$$

10. $$\frac{\partial^2 R}{\partial \omega^2}(\delta\omega)_4 = -\frac{\mu J_n J_\ell}{2 e \eta^2 a^{n+\ell+1}} \sum_q \sum_s q^2 B_{n,q} B_{\ell,s} \left(\overline{A_n D'_{2\ell}} - \bar{A}_n \bar{D}'_{2\ell}\right)$$

$$\overline{A_n D'_{2\ell}} = \frac{\sin(\beta+\alpha)}{2}\left[a_{n+\ell-2,s+1+q} - a_{n+\ell-2,s-1+q}\right] + \frac{\sin(\beta-\alpha)}{2}\left[a_{n+\ell-2,s+1-q} - a_{n+\ell-2,s-1-q}\right]$$

$$\bar{A}_n \bar{D}'_{2\ell} = \sin\beta\cos\alpha\left[a_{\ell-3,s+1} - a_{\ell-3,s-1}\right] a_{n-1,q} \quad \text{for } \ell \geq 3$$

$$\bar{A}_n \bar{D}'_{2\ell} = \frac{\sin\beta}{\eta^2}\cos\alpha\,(a_{n-1,q})\left[\overline{\cos(s+1)f} - \overline{\cos(s-1)f} + \frac{e}{2}\left\{\overline{\cos(s+2)f} - \overline{\cos(s-2)f}\right\}\right] \quad \text{for } \ell = 2$$

11. $$\overline{\frac{\partial^2 R}{\partial\omega\,\partial\eta}(\delta\eta)_1} = \frac{-\mu J_n J_\ell}{e\,\eta^{2\ell-3} a^{n+\ell+1}} \sum_q \sum_s q\, b'_{\ell-1,s} B_{n,q} B_{kp} \cos\beta \left(\overline{E_n B_{kp}}\right)$$

$$\overline{E_n B_{kp}} = \sum_{d=1}^{3} c(d)\left\{\sum_{t=1}^{t_1} r_t\, a_{\nu(d),\phi(d)} - \sum_{t=1}^{t_2} r_t\, a_{\nu(d),\rho(d)}\right\}\sin\alpha$$

The parameters for $\overline{E_n B_{kp}}$ are given in Table 4.

TABLE 4 (Parameters for $\overline{E_n B_{kp}}$)

d	c(d)	ν(d)	φ(d)	t_1	ρ(d)	t_2
1	$-\frac{e}{\eta}\left(\frac{n+1}{2}\right)$	n	$q+1-t$	$n+q+1$	$q+1+t$	$n-q-1$
2	$\frac{e}{\eta}\left(\frac{n+1}{2}\right)$	n	$q-1-t$	$n+q-1$	$q-1+t$	$n-q+1$
3	$-q\eta$	$n+1$	$q-t$	$n+q+1$	$q+t$	$n-q+1$

$$12.\ \overline{\frac{\partial^2 R}{\partial \omega \partial M}(\delta M)_2} = \frac{-\mu J_n J_\ell}{e\,\eta^{2\ell-3}\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} c_k\, q\, b'_{\ell-1,k}\, B_{n,q}\, B_{\ell,s}\left(\overline{E_n C'_\ell}\right)$$

$$\overline{E_n C'_\ell} = \sum_{d=1}^{3} \frac{c(d)}{2} \left\{ \frac{\sin(\beta+\alpha)}{s+k} a_{\nu(d),\phi(d)} + \frac{\sin(\beta-\alpha)}{s+k} a_{\nu(d),\sigma(d)} + \frac{\sin(\beta+\alpha)}{s-k} a_{\nu(d),\rho(d)} + \frac{\sin(\beta-\alpha)}{s-k} a_{\nu(d),\tau(d)} \right\}$$

The parameters for $\overline{E_n C'_\ell}$ are given in Table 5.

TABLE 5 (Parameters for $\overline{E_n C_\ell}$, $c(d)$ same as Table 4)

d	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	$s+k+q+1$	$s-k+q+1$	$s+k-q-1$	$s-k-q-1$
2	$s+k+q-1$	$s-k+q-1$	$s+k-q+1$	$s-k-q+1$
3	$s+k+q$	$s-k+q$	$s+k-q$	$s-k-q$

$$13.\ \overline{\frac{\partial^2 R}{\partial \omega \partial M}(\delta M)_3} = \frac{-\mu J_n J_\ell\, \eta}{e\, a^{n+\ell+1}} \sum_q \sum_s q\, B_{n,q}\, B_{\ell,s}\left(\overline{E_n D'_{12}}\right)$$

$$\overline{E_n D'_{12}} = \sum_{d=1}^{3} \frac{c(d)}{4} \left\{ \sin(\beta+\alpha)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\rho(d)}\right] + \sin(\beta-\alpha)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \right\}$$

The parameters for $\overline{E_n D'_{12}}$ are given in Table 6.

TABLE 6 (Parameters for $\overline{E_n D'_{12}}$, $c(d)$ same as Table 4)

d	$\nu(d)$	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	$n+\ell$	$s+q+2$	$s+q$	$s-q$	$s-q+2$
2	$n+\ell$	$s+q$	$s+q-2$	$s-q-2$	$s-q$
3	$n+\ell-1$	$s+q+1$	$s+q-1$	$s-q+1$	$s-q-1$

14. $$\overline{\frac{\partial^2 R}{\partial \omega \partial M}(\delta M)_4} = \frac{-\mu J_n J_\ell}{e \eta a^{n+\ell+1}} \sum_q \sum_s q B_{n,q} B_{\ell,s} \left(\overline{E_n D'_{2\ell}}\right)$$

$$\overline{E_n D'_{2\ell}} = \sum_{d=1}^{3} \frac{c(d)}{4} \Big\{ \sin(\beta+\alpha)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\rho(d)}\right] + \sin(\beta-\alpha)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \Big\}$$

The parameters $c(d)$, $\phi(d)$, $\rho(d)$, $\sigma(d)$, $\tau(d)$, are the same as in Table 5. The parameter $\nu(d)$ is given in Table 6.

TABLE 6 (Parameters for $\overline{E_n D'_{2\ell}}$)

d	$\nu(d)$
1	$n+\ell-1$
2	$n+\ell-1$
3	$n+\ell-2$

15. $$\overline{\frac{\partial R}{\partial \omega} \frac{\delta a}{a}} = \frac{-2\mu J_n J_\ell}{a^{n+\ell+1}} \sum_q \sum_s q B_{n,q} B_{\ell,s} \left(\overline{A'_n A_\ell} - \overline{A'_n}\,\overline{A_\ell}\right)$$

$$\overline{A'_n A_\ell} - \overline{A'_n}\,\overline{A_\ell} = \frac{\sin(\alpha+\beta)}{2}\left[a_{n+\ell,q+s} - a_{\ell-1,s}\, a_{n-1,q}\right] + \frac{\sin(\alpha-\beta)}{2}\left[a_{n+\ell,q-s} - a_{\ell-1,s}\, a_{n-1,q}\right]$$

16. $$\overline{\frac{\partial R}{\partial \omega}(\delta e)_1} = \frac{-\eta^2 \mu J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s q B_{n,q} B_{\ell,s} \left(\overline{A'_n A_\ell} - \overline{A'_n}\,\overline{A_\ell}\right)$$

The quantity $\overline{A'_n A_\ell} - \overline{A'_n}\,\overline{A_\ell}$ is defined in Equation (15) above.

17. $$\overline{\frac{\partial R}{\partial \omega}(\delta e)_2} = \frac{-\eta \mu J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s q\, s\, a_{\ell-1,s} B_{n,q} B_{\ell,s} \sin\beta \left(\overline{A'_n B_{KP}}\right)$$

The quantity $\overline{A'_n B_{KP}}$ is defined in Equation (5) above.

$$18. \quad \overline{\frac{\partial R}{\partial \omega}(\delta e)_2} = \frac{\eta \mu J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, q\, s\, a_{\ell-1,k}\, B_{n,q}\, B_{\ell,s} \left(\overline{A'_n C_\ell - \bar{A}'_n \bar{C}_\ell}\right)$$

The quantity $\overline{A'_n C_\ell - \bar{A}'_n \bar{C}_\ell}$ is defined in Equation (6) above.

$$19. \quad \overline{\frac{\partial R}{\partial \omega}(\delta i)_1} = \frac{\mu J_n J_\ell \cos i}{\eta\, a^{n+\ell+1} \sin i} \sum_q \sum_s q\, s\, a_{\ell-1,s}\, B_{n,q}\, B_{\ell,s} \sin\beta \left(\overline{A'_n B_{K\beta}}\right)$$

The quantity $\overline{A'_n B_{K\beta}}$ is defined in Equation (5) above.

$$20. \quad \overline{\frac{\partial R}{\partial \omega}(\delta i)_2} = \frac{-\mu J_n J_\ell \cos i}{\eta\, a^{n+\ell+1} \sin i} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, q\, s\, a_{\ell-1,k}\, B_{n,q}\, B_{\ell,s} \left(\overline{A'_n C_\ell - \bar{A}'_n \bar{C}_\ell}\right)$$

The quantity $\overline{A'_n C_\ell - \bar{A}'_n \bar{C}_\ell}$ is defined in Equation (6) above.

The long period terms for the inclination equation are the sum of the 20 equations given above.

The Eccentricity Equation

The expansion for the eccentricity equation is given by the expression,

$$\left[\frac{de}{dt}\right] = \left[f_{21} \left\{\left[\frac{\partial R}{\partial M}\right]\right\} + \frac{\partial R}{\partial M}\left\{ a \frac{\partial f_{21}}{\partial a}\frac{\delta a}{a} + \frac{\partial f_{21}}{\partial e}\delta e \right\} - f_{22}\left\{\left[\frac{\partial R}{\partial \omega}\right]\right\} \right.$$

$$\left. - \frac{\partial R}{\partial \omega}\left\{ a \frac{\partial f_{22}}{\partial a}\frac{\delta a}{a} + \frac{\partial f_{22}}{\partial e}\delta e \right\} \right]$$

Where $\left\{\left[\frac{\partial R}{\partial M}\right]\right\}$ represents the expansion,

$$\left\{\left[\frac{\partial R}{\partial M}\right]\right\} = a\frac{\partial^2 R}{\partial M \partial a}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial M \partial e}\delta e + \cdots + \frac{\partial^2 R}{\partial M^2}\delta M$$, and similarly

$$\left\{\left[\frac{\partial R}{\partial \omega}\right]\right\} = a\frac{\partial^2 R}{\partial \omega \partial a}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial \omega \partial e}\delta e + \cdots \quad \frac{\partial^2 R}{\partial \omega \partial M}\delta M \quad .$$

$$f_{21} = \frac{\eta^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$a\frac{\partial f_{21}}{\partial a} = \frac{-\eta^2}{2\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$\frac{\partial f_{21}}{\partial e} = -\frac{(1+e^2)}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$f_{22} = \frac{-\eta}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$a\frac{\partial f_{22}}{\partial a} = \frac{\eta}{2\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$\frac{\partial f_{22}}{\partial e} = -\frac{1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e^2 \eta}$$

It can be shown by an extension of the method of Kozai, Reference 4, that no long period coupling terms arise from $\left\{\left[\frac{\partial R}{\partial M}\right]\right\}$ and the product $\frac{\delta a}{a}\frac{\partial R}{\partial M}$. Below we give the long period terms for the products $\frac{\partial R}{\partial M}(\delta e)_2$ and $\frac{\partial R}{\partial q}(\delta e)_3$ since the remaining products of $\frac{de}{dt}$ have been given in the inclination equation.

$$1. \quad \overline{\frac{\partial R}{\partial M}(\delta e)_1} = \frac{\mu \eta J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s s\, a_{\ell-1,s}\, B_{n,q}\, B_{\ell,s} \sin\beta \left(\overline{E'_n B_{KP}}\right)$$

To find $\overline{E'_n B_{KP}}$ simply replace $\sin\alpha$ in Equation(11) of the inclination equation by $-\cos\alpha$.

$$2. \quad \overline{\frac{\partial R}{\partial M}(\delta e)_3} = \frac{-\mu \eta J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s \sum_{k=1}^{(i)} \epsilon_k\, s\, a_{\ell-1,k}\, B_{n,q}\, B_{\ell,s} \left(\overline{E'_n C_\ell}\right)$$

To find $\overline{E'_n C_\ell}$ replace $\sin(\beta-\alpha)$ in Equation (12) of the inclination equation by $-\sin(\beta-\alpha)$.

The Equation For the Longitude of the Node

The expansion for the node corresponding to Equation(3) is given by

$$\left[\frac{d\Omega}{dt}\right] = \left[f_5 \left\{ a\frac{\partial^2 R}{\partial i\,\partial a}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial i\,\partial e}\delta e + \frac{\partial^2 R}{\partial i^2}\delta i + \frac{\partial^2 R}{\partial i\,\partial \omega}\delta\omega \right.\right.$$

$$\left.\left. + \frac{\partial^2 R}{\partial i\,\partial H}\delta H \right\} + \frac{\partial R}{\partial i}\left\{ a\frac{\partial f_5}{\partial a}\frac{\delta a}{a} + \frac{\partial f_5}{\partial e}\delta e + \frac{\partial f_5}{\partial i}\delta i \right\} \right]$$

$$f_5 = \frac{1}{\mu^{1/2} a^{1/2} \eta \sin i}$$

$$a\frac{\partial f_5}{\partial a} = \frac{-1}{2\mu^{1/2} a^{1/2} \eta \sin i}$$

$$\frac{\partial f_5}{\partial e} = \frac{e}{\mu^{1/2} a^{1/2} \eta^3 \sin i}$$

$$\frac{\partial f_5}{\partial i} = \frac{-\cos i}{\mu^{1/2} a^{1/2} \eta \sin^2 i}$$

When the products are formed it will be seen below that each of the long period factors of $\left[\frac{d\Omega}{dt}\right]$ are the conjugates of the corresponding long period factors of $\left[\frac{di}{dt}\right]$. Consequently it is only necessary to replace each α of the Equations 1 through 6, and 15 through 20, of $\left[\frac{di}{dt}\right]$ by $\alpha + \frac{\pi}{2}$, and each α in equations 7 through 14, of $\left[\frac{di}{dt}\right]$ by $\alpha - \frac{\pi}{2}$, to obtain the corresponding equations for $\left[\frac{d\Omega}{dt}\right]$.

1. $$\overline{a\frac{\partial^2 R}{\partial i\,\partial a}\frac{\delta a}{a}} = \frac{-2\mu(n+1)J_n J_\ell \cos i}{a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} \left(\overline{A_n A_\ell} - \bar{A}_n \bar{A}_\ell\right)$$

2. $$\overline{\frac{\partial^2 R}{\partial i\,\partial e}}(\delta e)_1 = \frac{\mu \eta^2 J_n J_\ell \cos i}{e\, a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} \left(\overline{F_n A_\ell} - \bar{F}_n \bar{A}_\ell\right)$$

3. $$\frac{\partial^2 R}{\partial i\,\partial e}(\delta e)_2 = \frac{\mu\,\eta\,J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s s\, a_{\ell-1,s}\, B'_{n,q}\, B_{\ell,s} \sin\beta\, (\overline{F_n B_{KP}})$$

4. $$\frac{\partial^2 R}{\partial i\,\partial e}(\delta e)_3 = \frac{-\mu\,\eta\,J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, s\, a_{\ell-1,k}\, B'_{n,q}\, B_{\ell,s} (\overline{F_n C_\ell} - \bar{F}_n \bar{C}_\ell)$$

5. $$\frac{\partial^2 R}{\partial i^2}(\delta i)_1 = \frac{-\mu\,J_n J_\ell \cos i}{\eta\,a^{n+\ell+1} \sin i} \sum_q \sum_s s\, a_{\ell-1,s} (-\sin i\, B'_{n,q} + \cos^2 i\, B'_{n,q}) B_{\ell,s} \sin\beta\, (\overline{A_n B_{KP}})$$

6. $$\frac{\partial^2 R}{\partial i^2}(\delta i)_2 = \frac{\mu\,J_n J_\ell \cos i}{\eta\,a^{n+\ell+1} \sin i} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, s\, a_{\ell-1,k} (-\sin i\, B'_{n,q} + \cos^2 i\, B'_{n,q}) B_{\ell,s} (\overline{A_n C_\ell} - \bar{A}_n \bar{C}_\ell)$$

7. $$\frac{\partial^2 R}{\partial i\,\partial \omega}(\delta\omega)_1 = \frac{-\mu\,J_n J_\ell \cos i}{\eta^{2\ell}\,a^{n+\ell+1}} \sum_q \sum_s q\, B'_{n,q}\, W \cos\beta\, (\overline{A'_n B_{KP}})$$

8. $$\frac{\partial^2 R}{\partial i\,\partial \omega}(\delta\omega)_2 = \frac{-\mu\,J_n J_\ell \cos i}{\eta^{2\ell}\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, q\, B'_{n,q}\, W (\overline{A'_n C'_\ell} - \bar{A}'_n \bar{C}'_\ell)$$

9. $$\frac{\partial^2 R}{\partial i\,\partial \omega}(\delta\omega)_3 = \frac{-\mu\,J_n J_\ell \cos i}{2\,e\,a^{n+\ell+1}} \sum_q \sum_s q\, B'_{n,q}\, B_{\ell,s} (\overline{A'_n D'_{1\ell}} - \bar{A}'_n \bar{D}'_{1\ell})$$

10. $$\frac{\partial^2 R}{\partial i\,\partial \omega}(\delta\omega)_4 = \frac{-\mu\,J_n J_\ell \cos i}{2\,e\,\eta^2\,a^{n+\ell+1}} \sum_q \sum_s q\, B'_{n,q}\, B_{\ell,s} (\overline{A'_n D'_{2\ell}} - \bar{A}'_n \bar{D}'_{2\ell})$$

11. $$\frac{\partial^2 R}{\partial i\,\partial M}(\delta M)_1 = \frac{-\mu\,J_n J_\ell \cos i}{e\,\eta^{2\ell-3}\,a^{n+\ell+1}} \sum_q \sum_s b'_{\ell-1,s}\, B'_{n,q}\, B_{\ell,s} \cos\beta\, (\overline{E'_n B_{KP}})$$

12. $$\frac{\partial^2 R}{\partial i\,\partial M}(\delta M)_2 = \frac{-\mu\,J_n J_\ell \cos i}{e\,\eta^{2\ell-3}\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, b'_{\ell-1,k}\, B'_{n,q}\, B_{\ell,s} (\overline{E'_n C'_\ell})$$

$$13.\quad \overline{\frac{\partial^2 R}{\partial i\,\partial M}(\delta M)_3} = \frac{\mu\,\eta\,J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} \overline{(E'_n D'_\ell)}$$

$$14.\quad \overline{\frac{\partial^2 R}{\partial i\,\partial M}(\delta M)_4} = \frac{-\mu\,J_n J_\ell \cos i}{e\,\eta\,a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} (E'_n D'_{\ell\ell})$$

$$15.\quad \overline{\frac{\partial R}{\partial i}\frac{\delta a}{a}} = \frac{2\mu\,J_n J_\ell \cos i}{a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} (\overline{A'_n A_\ell} - \overline{A'_n \bar{A}_\ell})$$

$$16.\quad \overline{\frac{\partial R}{\partial i}(\delta e)_1} = \frac{\mu\,\eta^2 J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s B'_{n,q} B_{\ell,s} (\overline{A'_n A_\ell} - \overline{A'_n \bar{A}_\ell})$$

$$17.\quad \frac{\partial R}{\partial i}(\delta e)_2 = \frac{\mu\,\eta\,J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s s\,a_{\ell-1,s} B'_{n,q} B_{\ell,s} \sin\beta\,\overline{(A'_n B_{KP})}$$

$$18.\quad \frac{\partial R}{\partial i}(\delta e)_3 = \frac{-\mu\,\eta\,J_n J_\ell \cos i}{e\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, s\, a_{\ell-1,k} B'_{n,q} B_{\ell,s} (\overline{A'_n C_\ell} - \overline{A'_n \bar{C}_\ell})$$

$$19.\quad \frac{\partial R}{\partial i}(\delta i)_1 = \frac{-\mu\,J_n J_\ell \cos^2 i}{\eta\,a^{n+\ell+1}\sin i} \sum_q \sum_s s\,a_{\ell-1,s} B'_{n,q} B_{\ell,s} \overline{(A'_n B_{KP})}$$

$$20.\quad \frac{\partial R}{\partial i}(\delta i)_2 = \frac{\mu\,J_n J_\ell \cos^2 i}{\eta\,a^{n+\ell+1}\sin i} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, a_{\ell-1,k} - s\, B'_{n,q} B_{\ell,s} (\overline{A'_n C_\ell} - \overline{A'_n \bar{C}_\ell})$$

The long period terms for the equation of the node are the sum of the above 20 equations.

The Argument of Perigee Equation

The expansion for the argument of perigee equation corresponding to Equation (3) is given by

$$\left[\frac{d\omega}{dt}\right] = \Big[f_{\psi 1} \left\{\left[\frac{\partial R}{\partial i}\right]\right\} + \frac{\partial R}{\partial i}\left\{ a\frac{\partial f_{\psi 1}}{\partial a}\frac{\delta a}{a} + \frac{\partial f_{\psi 1}}{\partial e}\delta e + \frac{\partial f_{\psi 1}}{\partial i}\delta i \right\}$$

$$+ f_{\psi 2}\left\{ a\frac{\partial^2 R}{\partial e\,\partial a}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial e^2}\delta e + \frac{\partial^2 R}{\partial e\,\partial i}\delta i + \frac{\partial^2 R}{\partial e\,\partial\omega}\delta\omega + \frac{\partial^2 R}{\partial e\,\partial M}\delta M \right\}$$

$$+ \frac{\partial R}{\partial e}\left\{ a\frac{\partial f_{\psi 2}}{\partial a}\frac{\delta a}{a} + \frac{\partial f_{\psi 2}}{\partial e}\delta e \right\}\Big]$$

$$f_{\psi 1} = -\frac{\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin i} \qquad f_{\psi 2} = \frac{\eta}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$a\frac{\partial f_{\psi 1}}{\partial a} = \frac{\cos i}{2\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin i} \qquad a\frac{\partial f_{\psi 2}}{\partial a} = -\frac{\eta}{2\mu^{\frac{1}{2}} a^{\frac{1}{2}} e}$$

$$\frac{\partial f_{\psi 1}}{\partial e} = -\frac{e\cos i}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta^3 \sin i} \qquad \frac{\partial f_{\psi 2}}{\partial e} = \frac{-1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e^2 \eta}$$

$$\frac{\partial f_{\psi 1}}{\partial i} = \frac{1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} \eta \sin^2 i}$$

The long period terms for the expansion $\left\{\left[\frac{\partial R}{\partial i}\right]\right\}$, where

$$\left\{\left[\frac{\partial R}{\partial i}\right]\right\} = a\frac{\partial^2 R}{\partial i\,\partial a}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial i\,\partial e}\delta e + \frac{\partial^2 R}{\partial i^2}\delta i + \frac{\partial^2 R}{\partial i\,\partial\omega}\delta\omega$$

$$+ \frac{\partial^2 R}{\partial i\,\partial M}\delta M ,$$

as well as the long period terms of the functions $\frac{\partial R}{\partial a}\frac{\delta a}{a}$, $\frac{\partial R}{\partial e}\delta e$, $\frac{\partial R}{\partial i}\delta i$ are given in the equation for the node. In addition since the long period terms $\overline{(F_n' A_\ell - F_n' \bar{A}_\ell)}$, $\overline{F_n' B_{kp}}$, $\overline{F_n' C_\ell - F_n' \bar{C}_\ell}$, are given in the inclination equation, the corresponding terms of the perigee equation may be derived by replacing α in the above terms of the inclination equation by $\alpha + \frac{\pi}{2}$.

$$1.\ \frac{a\,\partial^2 R}{\partial e\,\partial a}\,\frac{\delta a}{a} = \frac{-2\mu(n+1)J_n J_l}{a^{n+l+1}} \sum_q \sum_s B_{n,q} B_{l,s} \left(\overline{F_n A_l} - \bar{F}_n \bar{A}_l\right)$$

$$2.\ \frac{\partial^2 R}{\partial e^2}(\delta e)_1 = \frac{\mu\,\eta^2 J_n J_l}{e\,a^{n+l+1}} \sum_q \sum_s B_{n,q} B_{l,s} \left(\overline{H_n A_l} - \bar{H}_n \bar{A}_l\right)$$

$$\overline{H_n A_l} = \sum_{d=1}^{9} \frac{c(d)}{2} \left\{ \cos(\alpha-\beta)\, a_{\nu(d),\delta(d)} + \cos(\alpha+\beta)\, a_{\nu(d),\rho(d)} \right\}$$

The parameters for $\overline{H_n A_l}$ are defined in Table 7.

TABLE 7 (Parameters for $\overline{H_n A_l}$)

d	$c(d)$	$\nu(d)$	$\delta(d)$	$\rho(d)$
1	$\frac{(n+q+1)(n+q+3)}{4}$	$n+l+2$	$q-s+2$	$q+s+2$
2	$\frac{2nq+2q^2+4q+n+1}{4\eta^2}$	$n+l+1$	$q-s+2$	$q+s+2$
3	$\frac{q(q+1)}{4\eta^2}$	$n+l$	$q-s+2$	$q+s+2$
4	$\frac{(n-q+1)(n-q+3)}{4}$	$n+l+2$	$q-s-2$	$q+s-2$
5	$\frac{n+1+2q^2-2nq+4q}{4\eta^2}$	$n+l+1$	$q-s-2$	$q+s-2$
6	$\frac{q(q-1)}{4\eta^2}$	$n+l$	$q-s-2$	$q+s-2$
7	$\frac{(n+q+1)(n-q+1)}{4}$	$n+l+2$	$q-s$	$q+s$
8	$-\frac{(q^2+n+1)}{2\eta^2}$	$n+l+1$	$q-s$	$q+s$
9	$-\frac{q^2}{2\eta^4}$	$n+l$	$q-s$	$q+s$

$$\bar{H}_n \bar{A}_l = \left\{ \frac{b''_{n-1,q}}{\eta^{2n-1}} - \frac{(2n-1)}{\eta}\left[\frac{a_{n-1,q}}{\eta}\left(\frac{e}{\eta} - 2n+1\right) + 2\,b_{n-1,q} \right] \right\} a_{l-1,s} \cos\alpha \cos\beta$$

$$3. \quad \frac{\partial^2 R}{\partial e^2}(\delta e)_n = \frac{\mu \eta J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s s\, a_{\ell-1,s}\, B_{n,q}\, \sin \beta\, (\overline{H\, B_{KP}})$$

$$\overline{H_n B_{KP}} = \sum_{d=1}^{9} c(d) \left\{ \sum_{t=1}^{t_1} a_{\nu(d),\phi(d)} - \sum_{t=1}^{t_2} a_{\nu(d),\gamma(d)} \right\} \sin \alpha$$

The parameters for $\overline{H_n B_{KP}}$ are defined in Table 8.

TABLE 8 (Parameters for $\overline{H_n B_{KP}}$, $c(d)$, $\nu(d)$, same as Table 7)

d	$\phi(d)$	t_1	$\gamma(d)$	t_2
1	$q-t+2$	$n+\ell+q+4$	$q+t+2$	$n+\ell-q$
2	$q-t+2$	$n+\ell+q+3$	$q+t+2$	$n+\ell-q-1$
3	$q-t+2$	$n+\ell+q+2$	$q+t+2$	$n+\ell-q-2$
4	$q-t-2$	$n+\ell+q$	$q+t-2$	$n+\ell-q+4$
5	$q-t-2$	$n+\ell+q-1$	$q+t-2$	$n+\ell-q+3$
6	$q-t-2$	$n+\ell+q-2$	$q+t-2$	$n+\ell-q+2$
7	$q-t$	$n+\ell+q+2$	$q+t$	$n+\ell-q+2$
8	$q-t$	$n+\ell+q+1$	$q+t$	$n+\ell-q+1$
9	$q-t$	$n+\ell+q$	$q+t$	$n+\ell-q$

$$4. \quad \frac{\partial^2 \ell}{\partial e^2}(\delta e)_s = \frac{-\mu \eta J_n J_\ell}{c\, a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, s\, a_{\ell-1,k}\, B_{n,q}\, B_{\ell,s}\, (\overline{H_n C_\ell} - \overline{H_n C_\ell})$$

$$\overline{H_n C_\ell} = \sum_{d=1}^{9} \frac{c(d)}{2} \left\{ \frac{\cos(\alpha-\beta)}{s+k} a_{\nu(d),\phi(d)} + \frac{\cos(\alpha+\beta)}{s+k} a_{\nu(d),\gamma(d)} + \frac{\cos(\alpha-\beta)}{s-k} a_{\nu(d),\sigma(d)} + \frac{\cos(\alpha+\beta)}{s-k} a_{\nu(d),\tau(d)} \right\}$$

The parameters for $\overline{H_n C_\ell}$ are defined in Table 9.

TABLE 9 (Parameters for $\overline{H_n C_\ell}$. $c(d)$ same as Table 7)

d	$\nu(d)$	$\phi(d)$	$\gamma(d)$	$\sigma(d)$	$\tau(d)$
1	$n+1$	$q-s-k+2$	$q+s+k+2$	$q-s+k+2$	$q+s-k+2$
2	n	$q-s-k+2$	$q+s+k+2$	$q-s+k+2$	$q+s-k+2$
3	$n-1$	$q-s-k+2$	$q+s+k+2$	$q-s+k+2$	$q+s-k+2$
4	$n+1$	$q-s-k-2$	$q+s+k-2$	$q-s+k-2$	$q+s-k-2$
5	n	$q-s-k-2$	$q+s+k-2$	$q-s+k-2$	$q+s-k-2$
6	$n-1$	$q-s-k-2$	$q+s+k-2$	$q-s+k-2$	$q+s-k-2$
7	$n+1$	$q-s-k$	$q+s+k$	$q-s+k$	$q+s-k$
8	n	$q-s-k$	$q+s+k$	$q-s+k$	$q+s-k$
9	$n-1$	$q-s-k$	$q+s+k$	$q-s+k$	$q+s-k$

$$\overline{H_n \bar{C}_\ell} = a_{\ell-1,k}(\gamma_{s+k} - \gamma_{s-k}) \left\{ \frac{b''_{n-1,q}}{\eta^{2n-1}} - \frac{(2n-1)}{n}\left[\frac{a_{n-1,q}}{n^2}\left(\frac{e}{n} - 2n+1\right) + 2 b_{n-1,q} \right] \right\} \cos k \cos \beta$$

5. $$\overline{\frac{\partial^2 R}{\partial e\, \partial i}(\delta i)_1} = \frac{-\mu J_n J_\ell \cos i}{\eta\, a^{n+\ell+1} \sin i} \sum_q \sum_s s\, a_{\ell+1,s}\, B'_{n,q}\, B_{\ell,s} \sin\beta\, (\overline{F_n B_{KP}})$$

6. $$\overline{\frac{\partial^2 R}{\partial e\, \partial i}(\delta i)_2} = \frac{\mu J_n J_\ell \cos i}{\eta\, a^{n+\ell+1} \sin i} \sum_q \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, s\, a_{\ell-1,k}\, B'_{n,q}\, B_{\ell,s} (\overline{F_n C_\ell} - \overline{F_n}\,\overline{C_\ell})$$

7. $$\overline{\frac{\partial^2 R}{\partial e\, \partial \omega}(\delta \omega)_1} = \frac{-\mu J_n J_\ell}{\eta^{2\ell} a^{n+\ell+1}} \sum_q \sum_s q\, B_{n,q}\, W \cos\beta\, (\overline{F'_n B_{KP}})$$

8. $$\overline{\frac{\partial^2 R}{\partial e\, \partial \omega}(\delta \omega)_2} = \frac{-\mu J_n J_\ell}{\eta^{2\ell} a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} q\, \epsilon_k\, B_{n,q}\, W (\overline{F'_n C'_\ell} - \overline{F'_n}\,\overline{C'_\ell})$$

9. $$\overline{\frac{\partial^2 R}{\partial e\, \partial \omega}(\delta \omega)_3} = \frac{-\mu J_n J_\ell}{2e\, a^{n+\ell+1}} \sum_q \sum_s q\, B_{n,q}\, B_{\ell,s} (\overline{F'_n D'_{1\ell}} - \overline{F'_n}\,\overline{D'_{1\ell}})$$

$$\overline{F'_n D'_{1\ell}} = \sum_{d=1}^{4} \frac{c(d)}{2} \left\{ \cos(\alpha-\beta)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\rho(d)}\right] + \cos(\alpha+\beta)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \right\}$$

The parameters for $\overline{F'_n D'_{1\ell}}$ are defined in Table 10.

TABLE 10 (Parameters for $\overline{F'_n D'_{1\ell}}$)

d	$c(d)$	$\nu(d)$	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	$\frac{n+1+q}{2}$	$n+\ell$	$q-s$	$q-s+2$	$q+s+2$	$q+s$
2	$\frac{n+1-q}{2}$	$n+\ell$	$q-s-2$	$q-s$	$q+s$	$q+s-2$
3	$\frac{q}{2\eta^2}$	$n+\ell-1$	$q-s$	$q-s+2$	$q+s+2$	$q+s$
4	$-\frac{q}{2\eta^2}$	$n+\ell-1$	$q-s-2$	$q-s$	$q+s$	$q+s-2$

$$\overline{F'_n}\;\overline{D'_{1\ell}} = a_{\ell-2,s+1}\left[b_{n-1,q} - \frac{(2n-1)}{\eta} a_{n-1,q}\right] \sin\alpha \sin\beta$$

10. $$\overline{\frac{\partial^2 R}{\partial e\, \partial \omega}(\delta \omega)_4} = \frac{-\mu J_n J_\ell}{2e\,\eta^2 a^{n+\ell+1}} \sum_q \sum_s q\, B_{n,q}\, B_{\ell,s} (\overline{F'_n D'_{2\ell}} - \overline{F'_n}\,\overline{D'_{2\ell}})$$

$$\overline{F'_n D'_{2\ell}} = \sum_{d=1}^{4} \frac{c(d)}{2} \left\{ \cos(\alpha-\beta)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\rho(d)}\right] + \cos(\alpha+\beta)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \right\}$$

$$\overline{F'_n} = \sin\alpha \left[b_{n+1,q} - \frac{(2n+1)}{\eta} a_{n+1,q} \right]$$

$$\overline{D'_{2l}} = \sin\beta \left[a_{l-3,s+1} - a_{l-3,s-1} \right] \quad \text{for } l \geq 3$$

$$\overline{D'_{2l}} = \frac{\sin\beta}{\eta^2} \left\{ \overline{\cos(s+1)f} - \overline{\cos(s-1)f} + \frac{e}{2}\left(\overline{\cos(s+2)f} - \overline{\cos(s-2)f} \right) \right\} \quad \text{for } l = 2$$

The definition of $\overline{\cos qf}$ is given in appendix 1. The parameters $v(d)$ for $\overline{F'_n D'_{2l}}$ are defined in Table 11. The remaining parameters of $\overline{F'_n D'_{2l}}$ are the same as those defined in Table 10.

TABLE 11 (Parameters $v(d)$ for $\overline{F'_n D'_{2l}}$)

d	$v(d)$
1	$n+l-1$
2	$n+l-1$
3	$n+l-2$
4	$n+l-2$

11. $$\overline{\frac{\partial^2 R}{\partial e \partial M}(\delta M)_1} = \frac{-\mu J_n J_l}{\eta^{2l-3} 2\, a^{n+l+1}} \sum_q \sum_s b'_{l-1,s} B_{n,q} B_{l,s} \cos\beta \left(\overline{G'_n B_{KP}} \right)$$

$$\overline{G'_n B_{KP}} = \sum_{d=1}^{10} c(d) \left\{ \sum_{t=1}^{t_1} r_t\, a_{v(d),\phi(d)} - \sum_{t=1}^{t_2} a_{v(d),f(d)} \right\} \cos\alpha$$

The parameters of $\overline{G'_n B_{KP}}$ are given in Table 12.

TABLE 12 (Parameters of $\overline{G'_n B_{KP}}$)

d	$c(d)$	$v(d)$	$\phi(d)$	t_1	$f(d)$	t_2
1	$-\frac{e(n+1+q)(n+2)}{4\eta}$	$n+1$	$q+t+2$	$n-q-1$	$q-t+2$	$n+q+3$
2	$-\frac{e(n+1)}{4\eta^3}$	n	$q+t+2$	$n-q-2$	$q-t+2$	$n+q+2$
3	$-\frac{\eta(n+1+q)(q+1)}{2}$	$n+2$	$q+t+1$	$n-q+1$	$q-t+1$	$n+q+3$
4	$-\frac{q(q+1)}{2\eta}$	$n+1$	$q+t+1$	$n-q$	$q-t+1$	$n+q+2$
5	$\frac{eq(n+2)}{2\eta}$	$n+1$	$q+t$	$n-q+1$	$q-t$	$n+q+1$
6	$\frac{eq(n+1)}{2\eta^3}$	n	$q+t$	$n-q$	$q-t$	$n+q$
7	$-\frac{\eta(n+1-q)(q-2)}{2}$	$n+2$	$q+t-1$	$n-q+3$	$q-t-1$	$n+q+1$
8	$\frac{q(q+1)}{2\eta}$	$n+1$	$q+t-1$	$n-q+2$	$q-t-1$	$n+q$
9	$\frac{e(n+1-q)(n+2)}{4\eta}$	$n+1$	$q+t-2$	$n-q+3$	$q-t-2$	$n+q-1$
10	$-\frac{e(n+1)q}{4\eta^3}$	n	$q+t-2$	$n-q+2$	$q-t-2$	$n+q-2$

12. $$\frac{\partial^2 R}{\partial e\,\partial M}(\delta M)_c = \frac{-\mu J_n J_\ell}{e\,\eta^{2\ell+3}\,a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell+1} \epsilon_k\, b'_{\ell+1,k}\, B_{n,q}\, B_{\ell,s}\, \left(\overline{G'_n C'_\ell}\right)$$

$$\overline{G'_n C'_\ell} = \sum_{d=1}^{10} \frac{c(d)}{2} \left\{ \frac{\cos(\kappa-\beta)}{s+k} a_{\nu(d),\phi(d)} - \frac{\cos(\kappa+\beta)}{s+k} a_{\nu(d),\rho(d)} + \frac{\cos(\kappa-\beta)}{s-k} a_{\nu(d),\sigma(d)} - \frac{\cos(\kappa+\beta)}{s-k} a_{\nu(d),\tau(d)} \right\}$$

The parameters of $\overline{G'_n C'_\ell}$ are given in Table 13.

TABLE 13 (Parameters of $\overline{G'_n C'_\ell}$. $c(d)$, $\nu(d)$, same as Table 11)

d	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	$q-s-k+2$	$q+s+k+2$	$q-s+k+2$	$q+s-k+2$
2	$q-s-k+2$	$q+s+k+2$	$q-s+k+2$	$q+s-k+2$
3	$q-s-k+1$	$q+s+k+1$	$q-s+k+1$	$q+s-k+1$
4	$q-s-k+1$	$q+s+k+1$	$q-s+k+1$	$q+s-k+1$
5	$q-s-k$	$q+s+k$	$q-s+k$	$q+s-k$
6	$q-s-k$	$q+s+k$	$q-s+k$	$q+s-k$
7	$q-s-k-1$	$q+s+k-1$	$q-s+k-1$	$q+s-k-1$
8	$q-s-k-1$	$q+s+k-1$	$q-s+k-1$	$q+s-k-1$
9	$q-s-k-2$	$q+s+k-2$	$q-s+k-2$	$q+s-k-2$
10	$q-s-k-2$	$q+s+k-2$	$q-s+k-2$	$q+s-k-2$

13. $$\frac{\partial^2 R}{\partial e\,\partial M}(\delta M)_3 = \frac{-\mu\,\eta\, J_n J_\ell}{c\, a^{n+\ell+1}} \sum_q \sum_s B_{n,q}\, B_{\ell,s}\, \left(\overline{G'_n D'_{1\ell}}\right)$$

$$\overline{G'_n D'_{1\ell}} = \sum_{d=1}^{10} \frac{c(d)}{4} \left\{ \cos(\kappa-\beta)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\rho(d)}\right] - \cos(\kappa+\beta)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \right.$$

The parameters of $\overline{G'_n D'_{1\ell}}$ are given in Table 14.

TABLE 14 (Parameters of $\overline{G'_n D'_{1\ell}}$. $c(d)$, same as Table 11)

d	$\nu(d)$	$\phi(d)$	$\rho(d)$	$\sigma(d)$	$\tau(d)$
1	$n+\ell+1$	$q-s+1$	$q-s+3$	$q+s+3$	$q+s+1$
2	$n+\ell$	$q-s+1$	$q-s+3$	$q+s+3$	$q+s+1$
3	$n+\ell+2$	$q-s$	$q-s+2$	$q+s+2$	$q+s$
4	$n+\ell+1$	$q-s$	$q-s+2$	$q+s+2$	$q+s$
5	$n+\ell+1$	$q-s-1$	$q-s+1$	$q+s+1$	$q+s-1$
6	$n+\ell$	$q-s-1$	$q-s+1$	$q+s+1$	$q+s-1$
7	$n+\ell+2$	$q-s-2$	$q-s$	$q+s$	$q+s-2$
8	$n+\ell+1$	$q-s-2$	$q-s$	$q+s$	$q+s-2$
9	$n+\ell+1$	$q-s-3$	$q-s-1$	$q+s-1$	$q+s-3$
10	$n+\ell+2$	$q-s-3$	$q-s-1$	$q+s-1$	$q+s-3$

14. $$\overline{\frac{\partial^2 R}{\partial e\, \partial M}(\delta M)_4} = \frac{-\mu J_n J_\ell}{e \eta a^{n+\ell+1}} \sum_q \sum_s B_{n,q} B_{\ell,s} (\overline{G'_n D'_{2\ell}})$$

$$\overline{G'_n D'_{2\ell}} = \sum_{d=1}^{10} \frac{c(d)}{4} \left\{ \cos(\alpha-\beta)\left[a_{\nu(d),\phi(d)} - a_{\nu(d),\gamma(d)}\right] - \cos(\alpha+\beta)\left[a_{\nu(d),\sigma(d)} - a_{\nu(d),\tau(d)}\right] \right\}$$

The parameters $\nu(d)$ for $\overline{F'_n D'_{2\ell}}$ are defined in Table 15. The remaining parameters of $\overline{F'_n D'_{2\ell}}$ are the same as those defined in Table 14.

TABLE 15 (Parameters $\nu(d)$ for $\overline{F'_n D'_{2\ell}}$)

d	$\nu(d)$
1	$n+\ell$
2	$n+\ell-1$
3	$n+\ell+1$
4	$n+\ell$
5	$n+\ell$
6	$n+\ell-1$
7	$n+\ell+1$
8	$n+\ell$
9	$n+\ell$
10	$n+\ell+1$

15. $$\overline{\frac{\partial R}{\partial e}\frac{\delta a}{a}} = \frac{2\mu J_n J_\ell}{a^{n+\ell+1}} \sum_q \sum_s B_{n,q} B_{\ell,s} (\overline{F_n A_\ell} - \overline{F_n A_\ell})$$

16. $$\overline{\frac{\partial R}{\partial e}(\delta e)_1} = \frac{\mu \eta^2 J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s B_{n,q} B_{\ell,s} (\overline{F_n A_\ell} - \overline{F_n \bar{A}_\ell})$$

17. $$\overline{\frac{\partial R}{\partial e}(\delta e)_2} = \frac{\mu \eta J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s s\, a_{\ell-1,s} B_{n,q} B_{\ell,s} \sin\beta\, (\overline{F_n B_{KP}})$$

18. $$\overline{\frac{\partial R}{\partial e}(\delta e)_3} = \frac{-\mu \eta J_n J_\ell}{e\, a^{n+\ell+1}} \sum_q \sum_s \sum_{k=0}^{\ell-1} e_k\, s\, a_{\ell-1,k} B_{n,q} B_{\ell,s} (\overline{F_n C_\ell} - \overline{F_n \tilde{C}_\ell})$$

The Equation For the Mean Anomaly

The expansion for the mean anomaly corresponding to Equation (3) is given by

$$\left[\frac{dM}{dt}\right] = \left[\frac{15}{8}\mu^{\frac{1}{2}} a^{-\frac{3}{2}} \left(\frac{\delta a}{a}\right)_m \left(\frac{\delta a}{a}\right)_l - f_{61}\left\{\left[\frac{\partial R}{\partial e}\right]\right\} - \frac{\partial R}{\partial e}\left\{a\frac{\partial f_{61}}{\partial a}\frac{\delta a}{a} + \frac{\partial f_{61}}{\partial e}\delta e\right\}\right.$$

$$- \frac{f_{62}}{a}\left\{a^2\frac{\partial^2 R}{\partial a^2}\frac{\delta a}{a} + a\frac{\partial^2 R}{\partial a\partial e}\delta e + a\frac{\partial^2 R}{\partial a\partial i}\delta i + a\frac{\partial^2 R}{\partial a\partial\omega}\delta\omega + a\frac{\partial^2 R}{\partial a\partial M}\delta M\right\}$$

$$\left. - a\frac{\partial R}{\partial a}\left\{\frac{\partial f_{62}}{\partial a}\frac{\delta a}{a}\right\}\right].$$

The first product $\left(\frac{\delta a}{a}\right)_m \left(\frac{\delta a}{a}\right)_l$ results from the expansion $a^{-\frac{3}{2}}$ in powers of $\frac{\delta a}{a}$. The quantities

$$\left\{\left[\frac{\partial R}{\partial e}\right]\right\} = a\frac{\partial^2 R}{\partial a\partial e}\frac{\delta a}{a} + \frac{\partial^2 R}{\partial e^2}\delta e + \frac{\partial^2 R}{\partial e\partial i}\delta i + \frac{\partial^2 R}{\partial e\partial\omega}\delta\omega + \frac{\partial^2 R}{\partial e\partial M}\delta M$$

$a\frac{\partial R}{\partial a}\frac{\delta a}{a}$, $\frac{\partial R}{\partial e}\delta e$, have been already defined in the argument of perigee equation. The remaining products of $\left[\frac{dM}{dt}\right]$ are given below. These products are the conjugates of the products given in $\left[\frac{di}{dt}\right]$. Thus the products in equations 1 through 6, and Equation (15), of $\left[\frac{dM}{dt}\right]$ are found by replacing α of the corresponding equations of $\left[\frac{di}{dt}\right]$ by $\alpha + \frac{\pi}{2}$. Similarly Equations 7 through 14 of $\left[\frac{dM}{dt}\right]$ are found by replacing α of the corresponding equations in $\left[\frac{di}{dt}\right]$ by $\alpha - \frac{\pi}{2}$.

$$f_{61} = \frac{\eta^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \qquad \frac{f_{62}}{a} = \frac{2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}}}$$

$$a\frac{\partial f_{61}}{\partial a} = \frac{-\eta^2}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e} \qquad \frac{\partial f_{62}}{\partial a} = \frac{1}{\mu^{\frac{1}{2}} a^{\frac{1}{2}}}$$

$$\frac{\partial f_{61}}{\partial e} = -\frac{(1+e^2)}{\mu^{\frac{1}{2}} a^{\frac{1}{2}} e^2}$$

1. $$\overline{a^2 \frac{\partial^2 R}{\partial a^2} \frac{\delta a}{a}} = \frac{2\mu(n+1)(n+2) J_n J_l}{a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} B_{n,q,p} B_{l,s} \left(\overline{A_n A_l} - \bar{A}_n \bar{A}_l \right)$$

2. $$\overline{a \frac{\partial^2 R}{\partial a \, \partial e} (\delta e)_1} = \frac{-\mu(n+1)\eta^2 J_n J_l}{e\, a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} B_{n,q,p} B_{l,s} \left(\overline{F_n A_l} - \bar{F}_n \bar{A}_l \right)$$

3. $$\overline{a \frac{\partial^2 R}{\partial a \, \partial e} (\delta e)_2} = \frac{-\mu(n+1)\eta J_n J_l}{e\, a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} 2 a_{l-1,s} B_{n,q,p} B_{l,s} \sin\beta \left(\overline{F_n B_{KP}} \right)$$

4. $$\overline{a \frac{\partial^2 R}{\partial a \, \partial e} (\delta e)_3} = \frac{\mu(n+1)\eta J_n J_l}{e\, a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} \sum_{k=0}^{l-1} \epsilon_k s\, a_{l-1,k} B_{n,q,p} B_{l,s} \left(\overline{F_n C_l} - \bar{F}_n \bar{C}_l \right)$$

5. $$\frac{a\, \partial^2 R}{\partial a \, \partial i} (\delta i)_1 = \frac{\mu(n+1) J_n J_l \cos^2 i}{\eta\, a^{n+l+1} \sin i} \sum_{\substack{q\\p}} \sum_{s} s\, a_{l-1,s} B'_{n,q,p} B_{l,s} \sin\beta \left(\overline{A_n B_{KP}} \right)$$

6. $$\frac{a\, \partial^2 R}{\partial a \, \partial i} (\delta i)_2 = -\frac{\mu(n+1) J_n J_l \cos^2 i}{\eta\, a^{n+l+1} \sin i} \sum_{\substack{q\\p}} \sum_{s} \sum_{k=0}^{l-1} \epsilon_k s\, a_{l-1,k} B'_{n,q,p} B_{l,s} \left(\overline{A_n C_l} - \bar{A}_n \bar{C}_l \right)$$

7. $$\frac{a\, \partial^2 R}{\partial a \, \partial \omega} (\delta\omega)_1 = \frac{\mu(n+1) J_n J_l}{\eta^{2l} a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} q\, B_{n,q,p} W \cos\beta \left(\overline{A'_n B_{KP}} \right)$$

8. $$\frac{a\, \partial^2 R}{\partial a \, \partial \omega} (\delta\omega)_2 = \frac{\mu(n+1) J_n J_l}{\eta^{2l} a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} \sum_{k=0}^{l-1} \epsilon_k q\, B_{n,q,p} W \left(\overline{A'_n C'_l} - \bar{A}'_n \bar{C}'_l \right)$$

9. $$\frac{a\, \partial^2 R}{\partial a \, \partial \omega} (\delta\omega)_3 = \frac{\mu(n+1) J_n J_l}{2e\, a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} q\, B_{n,q,p} B_{l,s} \left(\overline{A'_n D'_{1l}} - \bar{A}'_n \bar{D}'_{1l} \right)$$

10. $$\frac{a\, \partial^2 R}{\partial a \, \partial \omega} (\delta\omega)_4 = \frac{\mu(n+1) J_n J_l}{2e\, \eta^2 a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} q\, B_{n,q,p} B_{l,s} \left(\overline{A'_n D'_{2l}} - \bar{A}'_n \bar{D}'_{2l} \right)$$

11. $$\frac{a\, \partial^2 R}{\partial a \, \partial M} (\delta M) = \frac{\mu(n+1) J_n J_l}{e\, \eta^{2l-3} a^{n+l+1}} \sum_{\substack{q\\p}} \sum_{s} b'_{l-1,s} B_{n,q,p} B_{l,s} \cos\beta \left(\overline{E'_n B_{KP}} \right)$$

12. $$\overline{\frac{a\,\partial^2 R}{\partial a\,\partial M}(\delta M)_c} = \frac{\mu(n+1)\,J_n J_l}{\eta^{2l-3}\,e\,a^{n+l+1}} \sum_q \sum_s \sum_{k=0}^{l-1} \epsilon_k\, b'_{l-1,k}\, B_{n,q}\, B_{l,s}\, (\overline{E'_n C'_l})$$

13 $$\overline{\frac{a\,\partial^2 R}{\partial a\,\partial M}(\delta M)_s} = \frac{\mu\,\eta(n+1)\,J_n J_l}{e\,a^{n+l+1}} \sum_q \sum_s B_{n,q}\, B_{l,s}\, (\overline{E'_n D'_{1l}})$$

14. $$\overline{\frac{a\,\partial^2 R}{\partial a\,\partial M}(\delta M)_s} = \frac{\mu(n+1)\,J_n J_l}{e\,\eta\,a^{n+l+1}} \sum_q \sum_s B_{n,q}\, B_{l,s}\, (\overline{E'_n D'_{2l}})$$

15. $$\overline{\frac{a\,\partial R}{\partial a}\cdot\frac{\delta a}{a}} = \frac{-2\mu(n+1)\,J_n J_l}{a^{n+l+1}} \sum_q \sum_s B_{n,q}\, B_{l,s}\, (\overline{A_n A_l} - \bar{A}_n \bar{A}_l)$$

16. $$\overline{\left(\frac{\delta a}{a}\right)_n \left(\frac{\delta a}{a}\right)_l} = \frac{4\,J_n J_l}{a^{n+l+1}} \sum_q \sum_s B_{n,q}\, B_{l,s}\, (\overline{A_n A_l} - \bar{A}_n \bar{A}_l)$$

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Appendix 1

The Inclination Function

A convenient form of the inclination function is obtained from Reference 3.

$$B_{n,q} = \frac{1}{2^{n-1}} \cos\left[(q - ODQ)\frac{\pi}{2}\right] \epsilon_q \sum_{d=\frac{q-ODQ}{2}}^{\frac{n-ODQ}{2}} (-1)^{\frac{n-ODQ-2d}{2}} \quad \times$$

$$\binom{2d+q}{d-\frac{q-ODQ}{2}} \frac{(n+2d+ODQ)!\,(\sin i)^{2d+ODQ}}{\left(\frac{n+2d+ODQ}{2}\right)!\left(\frac{n-2d-ODQ}{2}\right)!\,(2d+ODQ)!\;2^{2d+ODQ}}$$

where $ODQ = 0$ for q even or zero

$ODQ = 1$ for q odd

$q = 2i + ODN \qquad i = 0, 1, 2, \ldots \frac{n-ODN}{2}$

$$B'_{n,q} = \frac{d\,B_{n,q}}{d(\sin i)}$$

$$B''_{n,q} = \frac{d^2 B_{n,q}}{d(\sin i)^2}$$

The Averaging Function

The averaging function arises by first considering the expansion for

$$\left(\frac{a}{r}\right)^{s} = \frac{2}{\eta^{2s}} \sum_{k=0}^{s} \epsilon_k \, a'_{s,k} \cos kf \qquad \begin{aligned} \epsilon_k &= \tfrac{1}{2} \text{ for } k=0, \\ &= 1 \text{ for } k>0 \end{aligned} \qquad (1)$$

$$a'_{s,k} = \sum_{d=0}^{\frac{s-k}{2}} \binom{2d+k}{d} \binom{s}{2d+k} \left(\frac{e}{2}\right)^{2d+k}$$

From the definition of the mean value, the function A is given by

$$\bar{A} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{p+1} \cos(\alpha + qf)\, dM$$

If we substitute Equation (1) and the expresson for dM in terms of df we find

$$\overline{\left(\frac{a}{r}\right)^{p+1} \cos(\alpha + qf)} = a_{p-1,q}$$

where

$$a_{p-1,q} = \frac{a'_{p-1,q}}{\eta^{2p-1}} \qquad \text{for } q \le p-1$$

$$a_{p-1,q} = 0 \qquad q > p-1$$

$$a_{p-1,-q} = a_{p-1,q}$$

$$a_{0,q} = 0 \qquad \text{for } q \neq 0$$

Expressions for the derivates of the coefficients with respect to e are also useful. We find

$$b'_{p-1,q} = \frac{d\,a'_{p-1,q}}{de} = \sum_{d=0}^{\frac{p-q-1}{2}} \frac{2d+q}{2} \binom{2d+q}{d} \binom{p-1}{2d+q} \left(\frac{e}{2}\right)^{2d+q-1}$$

$$b''_{p-1,q} = \frac{d^2 a'_{p-1,q}}{de^2} = \sum_{d=0}^{\frac{p-q-1}{2}} \frac{(2d+q)(2d+q-1)}{4} \binom{2d+q}{d} \binom{p-1}{2d+q} \left(\frac{e}{2}\right)^{2d+q-2}$$

$$b_{p-1,q} = \frac{b'_{p-1,q}}{\eta^{2p-1}}$$

$$\frac{d\,a_{p-1,q}}{de} = b_{p-1,q} - \frac{(2p-1)}{\eta} a_{p-1,q}$$

$$\frac{d^2 a_{p-1,q}}{de^2} = \frac{b''_{p-1,q}}{\eta^{2p-1}} - \frac{(2p-1)}{\eta} \left\{ \frac{a_{p-1,q}}{\eta} \left[\frac{e}{\eta} - (2p-1) \right] + 2\, b_{p-1,q} \right\}$$

The Equation of the Center

The expansion for the equation of the center (f-M) is given by Reference 1,

$$B_{KP} = f - M = -2 \sum_{p=1}^{\infty} \gamma_p \sin pf$$

where

$$\gamma_p = \frac{\overline{\cos pf}}{p}$$

$$\overline{\cos pf} = \left(\frac{-e}{1+\eta}\right)^p (1 + p\eta)$$

$$\eta = \sqrt{1-e^2}$$

Appendix 2

Symbols For Functions

$$A_l = \left(\frac{a}{r}\right)^{l+1} \cos(\beta + s f)$$

$$A_n = \left(\frac{a}{r}\right)^{n+1} \cos(\alpha + q f)$$

$$\bar{A}_l = a_{l+1,s} \cos\beta$$

$$\bar{A}_n = a_{n+1,q} \cos\alpha$$

$$B_{KP} = f - M$$

$$\bar{B}_{KP} = 0$$

$$C_l = \frac{\cos[\beta + (s+k) f]}{s+k} + \frac{\cos[\beta + (s-k) f]}{s-k} \qquad \text{for } s-k \geq 0$$

$$C_l = \frac{\cos(\beta + 2s f)}{2s} \qquad \text{for } k = s$$

$$\bar{C}_l = (\gamma_{s+k} + \gamma_{s-k}) \cos\beta \qquad \gamma_0 = 0$$

$$D'_{1l} = \left(\frac{a}{r}\right)^{l} \{ \sin[\beta + (s+1) f] - \sin[\beta + (s-1) f] \}$$

$$\bar{D}'_{1l} = (a_{l-2,s+1} - a_{l-2,s-1}) \sin\beta$$

$$D'_{2l} = \left(\frac{a}{r}\right)^{l-1} \{ \sin[\beta + (s+1) f] - \sin[\beta + (s-1) f] \}$$

$$\bar{D}'_{2l} = (a_{l-3,s+1} - a_{l-3,s-1}) \sin\beta \qquad \text{for } l \geq 3$$

$$\bar{D}'_{2l} = \left\{ \overline{\cos(s+1) f} - \overline{\cos(s-1) f} + \frac{e}{2}\left[\overline{\cos(s+2) f} - \overline{\cos(s-2) f}\right] \right\} \frac{\sin\beta}{\eta^2} \qquad \text{for } l = 2$$

$$E = -\frac{(n+1)}{2} \frac{e}{\eta} \left(\frac{a}{r}\right)^{n+2} \cos[\alpha + (q+1) f] + \left(\frac{n+1}{2}\right) \frac{e}{\eta} \left(\frac{a}{r}\right)^{n+2} \cos[\alpha + (q-1) f]$$
$$- q \eta \left(\frac{a}{r}\right)^{n+3} \cos(\alpha + q f)$$

$$\bar{E} = \left\{ -\left(\frac{n+1}{2}\right) \frac{e}{\eta} a_{n,q+1} + \left(\frac{n+1}{2}\right) \frac{e}{\eta} a_{n,q-1} - q \eta a_{n+1,q} \right\} \cos\alpha$$

$$F = \left\{ \left(\frac{n+1+q}{2\eta}\right)\left(\frac{a}{r}\right)^{n+2} + \frac{q}{2\eta^2}\left(\frac{a}{r}\right)^{n+1} \right\} \cos[\alpha + (q+1)f]$$

$$+ \left\{ \frac{(n+1-q)}{2\eta}\left(\frac{a}{r}\right)^{n+2} - \frac{q}{2\eta^2}\left(\frac{a}{r}\right)^{n+1} \right\} \cos[\alpha + (q-1)f]$$

$$\bar{F} = \left\{ \left(\frac{n+1+q}{2\eta}\right) a_{n,q+1} + \frac{q}{2\eta^2} a_{n-1,q+1} + \left(\frac{n+1-q}{2\eta}\right) a_{n,q-1} - \frac{q}{2\eta^2} a_{n-1,q-1} \right\} \cos\alpha$$

$$G' = \left\{ \frac{-e(n+1+q)(n+2)}{4\eta}\left(\frac{a}{r}\right)^{n+3} - \frac{e(n+1)q}{4\eta^3}\left(\frac{a}{r}\right)^{n+2} \right\} \sin[\alpha + (q+2)f]$$

$$+ \left\{ \frac{-\eta(n+1+q)(q+1)}{2}\left(\frac{a}{r}\right)^{n+4} - \frac{q(q+1)}{2\eta}\left(\frac{a}{r}\right)^{n+3} \right\} \sin[\alpha + (q+1)f]$$

$$+ \left\{ \frac{eq(n+2)}{2\eta}\left(\frac{a}{r}\right)^{n+3} + \frac{e(n+1)q}{2\eta^3}\left(\frac{a}{r}\right)^{n+2} \right\} \sin[\alpha + qf]$$

$$+ \left\{ \frac{-\eta(n+1-q)(q-2)}{2}\left(\frac{a}{r}\right)^{n+4} + \frac{q(q+1)}{2\eta}\left(\frac{a}{r}\right)^{n+3} \right\} \sin[\alpha + (q-1)f]$$

$$+ \left\{ \frac{e(n+2)(n+1-q)}{4\eta}\left(\frac{a}{r}\right)^{n+3} - \frac{eq(n+1)}{4\eta^3}\left(\frac{a}{r}\right)^{n+2} \right\} \sin[\alpha + (q-2)f]$$

$$\bar{G}' = 0$$

$$H = \left\{ \frac{(n+q+1)(n+q+3)}{4}\left(\frac{a}{r}\right)^{n+3} + \frac{(2nq+2q^2+4q+n+1)}{4\eta^2}\left(\frac{a}{r}\right)^{n+2} + \frac{q(q+1)}{4\eta^2}\left(\frac{a}{r}\right)^{n+1} \right\} \cos[\alpha + (q+2)f]$$

$$+ \left\{ \frac{(n-q+1)(n-q+3)}{4}\left(\frac{a}{r}\right)^{n+3} - \frac{(2nq-2q^2+4q-n-1)}{4\eta^2}\left(\frac{a}{r}\right)^{n+2} + \frac{q(q-1)}{4\eta^2}\left(\frac{a}{r}\right)^{n+1} \right\} \cos[\alpha + (q-2)f]$$

$$+ \left\{ \frac{(n+q+1)(n-q+1)}{4}\left(\frac{a}{r}\right)^{n+3} - \frac{(q^2+n+1)}{2\eta^2}\left(\frac{a}{r}\right)^{n+2} - \frac{q^2}{2\eta^2}\left(\frac{a}{r}\right)^{n+1} \right\} \cos(\alpha + qf)$$

$$\bar{H} = -\left\{ \frac{b''_{n-1,q}}{\eta} - \frac{(2n-1)}{\eta}\left[2 b_{n-1,q} + \frac{a_{n-1,q}}{n}\left\{ \frac{e}{\eta} - (2n-1) \right\} \right] \right\} \cos\alpha$$

$$W = \left[(2l-1) B_{l,s} - \frac{\cos^2 i}{\sin i} B'_{l,s} \right] a'_{l-1,s} + \frac{\eta^2}{e} b'_{l-1,s} B_{l,s}$$

Appendix 3

Derivatives of the Disturbing Function

$$R_n = \frac{\mu J_n}{a^{n+1}} \sum_q B_{n,q} (A - \bar{A})$$

$$\frac{\partial R_n}{\partial a} = -\frac{\mu (n+1) J_n}{a^{n+2}} \sum_q B_{n,q} (A - \bar{A})$$

$$\frac{\partial R_n}{\partial e} = \frac{\mu J_n}{a^{n+1}} \sum_q B_{n,q} (F - \bar{F})$$

$$\frac{\partial R_n}{\partial i} = \frac{\mu J_n \cos i}{a^{n+1}} \sum_q B'_{n,q} (A - \bar{A})$$

$$\frac{\partial R_n}{\partial \omega} = -\frac{\mu J_n}{a^{n+1}} \sum_q q\, B_{n,q} (A' - \bar{A}')$$

$$\frac{\partial R_n}{\partial M} = \frac{\mu J_n}{a^{n+1}} \sum_q B_{n,q} E'$$

$$\frac{\partial^2 R_n}{\partial a^2} = \frac{\mu (n+1)(n+2) J_n}{a^{n+3}} \sum_q B_{n,q} (A - \bar{A})$$

$$\frac{\partial^2 R_n}{\partial a \partial e} = -\frac{\mu (n+1) J_n}{a^{n+2}} \sum_q B_{n,q} (F - \bar{F})$$

$$\frac{\partial^2 R_n}{\partial a \partial i} = -\frac{\mu (n+1) J_n \cos i}{a^{n+2}} \sum_q B'_{n,q} (A - \bar{A})$$

$$\frac{\partial^2 R_n}{\partial a \partial \omega} = \frac{\mu (n+1) J_n}{a^{n+2}} \sum_q q\, B_{n,q} (A' - \bar{A}')$$

$$\frac{\partial^2 R}{\partial a \partial M} = -\frac{\mu (n+1) J_n}{a^{n+2}} \sum_q B_{n,q} E'$$

$$\frac{\partial^2 R_n}{\partial e \partial a} = -\frac{\mu (n+1) J_n}{a^{n+2}} \sum_q B_{n,q} (F - \bar{F})$$

$$\frac{\partial^2 R_n}{\partial e^2} = \frac{\mu J_n}{a^{n+1}} \sum_q B_{n,q} (H - \bar{H})$$

$$\frac{\partial^2 R_n}{\partial e \partial i} = \frac{\mu J_n \cos i}{a^{n+1}} \sum_q B'_{n,q} (F - \bar{F})$$

$$\frac{\partial^2 R_n}{\partial e \partial \omega} = -\frac{\mu J_n}{a^{n+1}} \sum_q q B_{n,q} (F' - \bar{F}')$$

$$\frac{\partial^2 R_n}{\partial e \partial M} = \frac{\mu J_n}{a^{n+1}} \sum_q B_{n,q} G'$$

$$\frac{\partial^2 R_n}{\partial i \partial a} = -\frac{\mu (n+1) J_n \cos i}{a^{n+2}} \sum_q B'_{n,q} (A - \bar{A})$$

$$\frac{\partial^2 R_n}{\partial i \partial e} = \frac{\mu J_n \cos i}{a^{n+1}} \sum_q B'_{n,q} (F - \bar{F})$$

$$\frac{\partial^2 R_n}{\partial i \partial \omega} = -\frac{\mu J_n \cos i}{a^{n+1}} \sum_q B'_{n,q} (A' - \bar{A}')$$

$$\frac{\partial^2 R_n}{\partial i \partial M} = \frac{\mu J_n \cos i}{a^{n+1}} \sum_q B'_{n,q} E'$$

$$\frac{\partial^2 R_n}{\partial \omega \partial a} = \frac{\mu (n+1) J_n}{a^{n+2}} \sum_q q B_{n,q} (A' - \bar{A}')$$

$$\frac{\partial^2 R}{\partial \omega \partial e} = -\frac{\mu J_n}{a^{n+1}} \sum_q q B_{n,q} (F - \bar{F})$$

$$\frac{\partial^2 R}{\partial \omega \partial i} = -\frac{\mu J_n \cos i}{a^{n+1}} \sum_q q B'_{n,q} (A' - \bar{A}')$$

$$\frac{\partial^2 R}{\partial \omega^2} = -\frac{\mu J_n}{a^{n+1}} \sum_q q^2 B_{n,q} (A - \bar{A})$$

$$\frac{\partial^2 R}{\partial \omega \partial M} = \frac{\mu J_n}{a^{n+1}} \sum_q q B_{n,q} E$$

Appendix 4

Perturbations in the Elements

$$\frac{\delta a}{a} = \frac{2 J_\ell}{a^\ell} \sum_s B_{\ell,s} (A_\ell - \bar{A}_\ell)$$

$$\delta e = (\delta e)_1 + (\delta e)_2 + (\delta e)_3$$

$$(\delta e)_1 = \frac{\eta^2 J_\ell}{e\, a^\ell} \sum_s B_{\ell,s} (A_\ell - \bar{A}_\ell)$$

$$(\delta e)_2 = \frac{\eta\, J_\ell}{e\, a^\ell} \sum_s s\, a_{\ell-1,s} B_{\ell,s} B_{KP} \sin\beta$$

$$(\delta e)_3 = -\frac{\eta\, J_\ell}{e\, a^\ell} \sum_s \sum_{k=0}^{\ell-1} s\, a_{\ell-1,k}\, \epsilon_k\, B_{\ell,s} (C_\ell - \bar{C}_\ell)$$

$$(\delta i) = (\delta i)_1 + (\delta i)_2$$

$$(\delta i)_1 = -\frac{J_\ell \cos i}{a^\ell \eta \sin i} \sum_s s\, a_{\ell-1,s} B_{\ell,s} B_{KP} \sin\beta$$

$$(\delta i)_2 = \frac{J_\ell \cos i}{a^\ell \eta \sin i} \sum_s \sum_{k=0}^{\ell-1} s\, a_{\ell-1,k}\, \epsilon_k\, B_{\ell,s} (C_\ell - \bar{C}_\ell)$$

$$(\delta \omega) = (\delta\omega)_1 + (\delta\omega)_2 + (\delta\omega)_3 + (\delta\omega)_4$$

$$(\delta\omega)_1 = \frac{J_\ell}{\eta^{2\ell} a^\ell} \sum_s W B_{KP} \cos\beta$$

$$(\delta\omega)_2 = \frac{J_\ell}{\eta^{2\ell} a^\ell} \sum_s \sum_{k=0}^{\ell-1} \epsilon_k W (C_\ell' - \bar{C}_\ell')$$

$$(\delta\omega)_3 = \frac{J_\ell}{2 e\, a^\ell} \sum_s B_{\ell,s} (D_{1\ell}' - \bar{D}_{1\ell}')$$

$$(\delta\omega)_4 = \frac{J_\ell}{2 e\, a^\ell} \sum_s B_{\ell,s} (D_{2\ell}' - \bar{D}_{2\ell}')$$

$$\delta M = (\delta M)_1 + (\delta M)_2 + (\delta M)_3 + (\delta M)_4$$

$$(\delta M)_1 = \frac{-J_\ell}{e\,\eta^{2\ell-3}\,a^\ell} \sum_s t'_{\ell-1,s}\, B_{\ell,s}\, \ell_{\kappa\nu} \cos\beta$$

$$(\delta M)_2 = \frac{-J_\ell}{e\,\eta^{2\ell-3}\,a^\ell} \sum_s \sum_{k=0}^{\ell-1} \epsilon_k\, t'_{\ell-1,k}\, B_{\ell,s}\, (C'_\ell - \overline{C}'_\ell)$$

$$(\delta M)_3 = \frac{-\eta\, J_\ell}{e\, a^\ell} \sum_s B_{\ell,s}\, (D'_{1\ell} - \overline{D}'_{1\ell})$$

$$(\delta M)_4 = \frac{-J_\ell}{e\,\eta\, a^\ell} \sum_s B_{\ell,s}\, (D'_{2\ell} - \overline{D}'_{2\ell})$$

Appendix 5

Some Useful Trigonometric Identities

$$f - M = -2 \sum_{t=1}^{\infty} \gamma_t \sin t f$$

$$\cos(\alpha + q f) \sin t f = \frac{1}{2}\left\{\sin[\alpha + (q+t) f] - \sin[\alpha + (q-t) f]\right\}$$

$$(f - M)\cos(\alpha + q f) = \sum_{t=1}^{\infty} \gamma_t \left\{\sin[\alpha + (q-t) f] - \sin[\alpha + (q+t) f]\right\}$$

$$(f - M)\sin(\alpha + q f) = \sum_{t=1}^{\infty} \gamma_t \left\{\cos[\alpha + (q+t) f] - \cos[\alpha + (q-t) f]\right\}$$

$$\sin(\alpha + q f)\cos(\beta + s f) = \frac{\sin[\alpha + \beta + (q+s) f]}{2} + \frac{\sin[\alpha - \beta + (q-s) f]}{2}$$

$$\cos(\alpha + q f)\sin(\beta + s f) = \frac{\sin[\beta + \alpha + (s+q) f]}{2} + \frac{\sin[\beta - \alpha + (s-q) f]}{2}$$

$$\sin(\alpha + q f)\sin(\beta + s f) = \frac{\cos[\alpha - \beta + (q-s) f]}{2} - \frac{\cos[\alpha + \beta + (q+s) f]}{2}$$

$$\cos(\alpha + q f)\cos(\beta + s f) = \frac{\cos[\alpha - \beta + (q-s) f]}{2} + \frac{\cos[\alpha + \beta + (q+s) f]}{2}$$

APPENDIX 6

NOMENCLATURE

Functions A_n, A_l, C_l, D'_{il}, D'_{jl}, E_n, F_n, G'_n, H_n are defined in Appendix 2. The primed function when not given explicitly is formed from the unprimed function. When the function has the subscript n then the primed function is formed by replacing α in the unprimed function by $\alpha - \frac{\pi}{2}$. Similarly, when the function has the subscript l the primed function is formed by replacing β in the unprimed function by $\beta - \frac{\pi}{2}$.

$B_{\gamma p}$ = f-M = equation of the center

$B_{n,q}$ = the inclination function

$B'_{n,q}$ the derivative of $B_{n,q}$ with respect to sine of i.

$B''_{n,q}$ the second derivative of $B_{n,q}$ with respect to sine(i).

J_n zonal harmonic of order n

M mean anomaly

ODL function which is zero for l even, and equals one for l odd.

ODN function which is zero for n even, and equals one for n odd.

R disturbing function

R_l disturbing function of l^{th} zonal harmonic

R_n disturbing function of n^{th} zonal harmonic

W function defined in Appendix 2

a semi major axis

$a'_{p-1,q}$ function defined in Appendix 1

$a_{p-1,q} = \frac{a'_{p-1,q}}{\eta^{2p-1}}$

$b'_{p-1,q} = \frac{d\, a_{p-1,q}}{de}$

$b''_{p-1,q} = \frac{d^2 a'_{p-1,q}}{de^2}$

e eccentricity

i inclination

i summation index (when it cannot be confused with the inclination)

j summation index

$q = 2i + ODN$, $i = 0, 1, 2, \ldots, \frac{n - ODN}{2}$

r radius vector of satellite

$s = 2j + ODL$, $j = 0, 1, 2, \ldots, \frac{l - ODL}{2}$

$\alpha = q\omega - ODN \frac{\pi}{2}$

$\beta = s\omega - ODL \frac{\pi}{2}$

γ_q coefficient appearing in equation of the center

$\epsilon_q = \frac{1}{2}$ for $q = 0$

$= 1$ for $q > 0$

$\eta = \sqrt{1 - e^2}$

$\gamma(d), \phi(d), \rho(d), \theta(d), \sigma(d), \tau(d)$ subscripts appearing in averaging functions.

ω argument of perigee

Ω right ascension of node